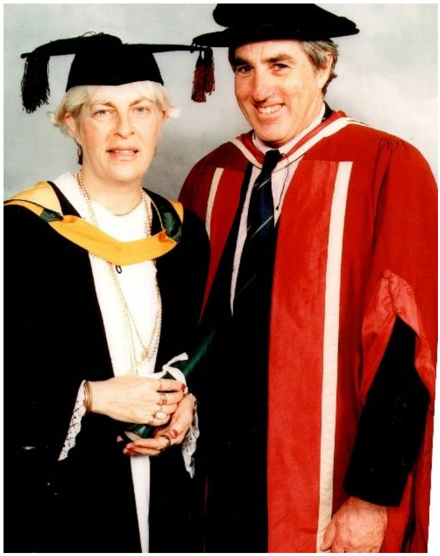
10. THE ALEXANDER GROUP OF A KNOT

§10.1. The Face Group

We're now ready for a graduate account of the Alexander Number. In fact we'll not just produce a number but a finite abelian group. The Alexander Number is simply the order (or size) of this group. However the Alexander Group is more discriminating and sometimes two knots with the same Alexander Number can be shown to be inequivalent because they have different Alexander Groups (although with the same size).

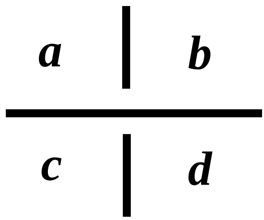


For example, if one knot had $\mathbb{Z}_9$ as its Alexander Group and another had $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ they would both have Alexander number 9 yet, having different Alexander Groups we could conclude that the knots are not equivalent.

Let K be a knot and let M be a map for it. We define an abelian group for the knot in terms of generators and relations as follows:

Assign a generator to each face (including the outside). In fact we'll use the name of each face as the name of the corresponding generator.

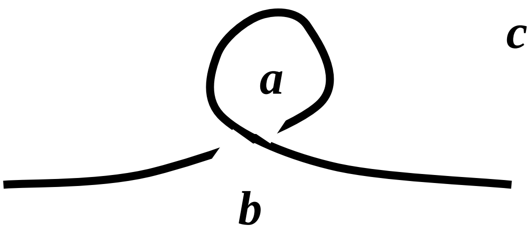
For each crossing



take the relation

$$\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{c} + \boldsymbol{d}$$

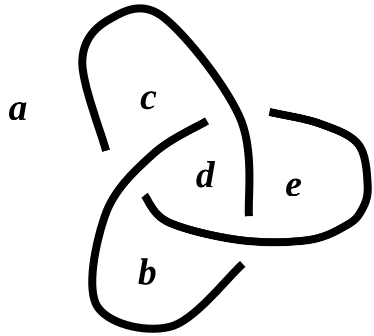
For a crossing of the form



this becomes $\boldsymbol{a} + \boldsymbol{c} = \boldsymbol{a} + \boldsymbol{b}$, whence $\boldsymbol{b} = \boldsymbol{c}$.

The abelian group with these generators and relations is called the Face Group of the map. Now if this group depended on the map, and not the knot itself, it wouldn't be a very useful concept. However it *does* only depend on the knot and so we call it the **Face Group F(K)** of the knot.

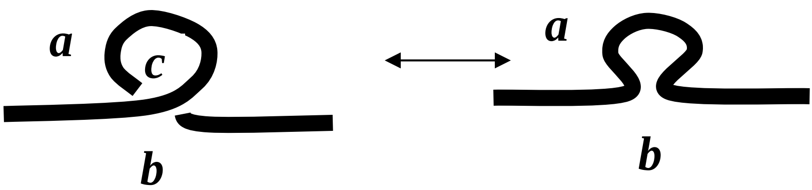
Example 1: F(trefoil) =
$= [\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d} \mid \mathbf{a} + \boldsymbol{c} = \boldsymbol{b} + \boldsymbol{d}, \boldsymbol{c} + \boldsymbol{d} = \boldsymbol{a} + \boldsymbol{e}, \boldsymbol{d} + \boldsymbol{e} = \boldsymbol{a} + \mathbf{b}]$



§10.2. The Face Group is an Invariant of a Knot

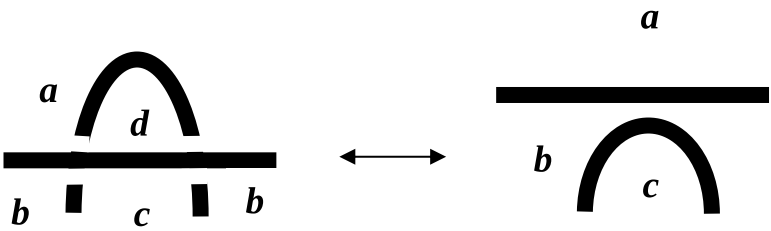
We'll now show that the three Reidemeister moves don't change the Face Group (up to isomorphism). Since any map of a knot can be transformed to any other equivalent map, by some sequence of Reidemeister moves, this shows that the Face Group depends only on the knot. In other words, it's an *invariant* of the knot.

Theorem 1: Each of the three Reidemeister moves leaves the Face Group unchanged, up to isomorphism.
Proof: Let K be a knot.
Type I moves don't change F(K):



The Face Groups will be the same except that the one on the left has an extra generator, ***c***, and an extra relation ***a*** + ***c*** = ***a*** + ***b***. A consequence of this relation is ***c*** = ***b***. Since ***c*** isn't involved in any other relation we may remove ***c*** as a generator, as well as relation ***c*** = ***b***. This group is then the Face Group of the knot on the right.

Type II moves don't change F(K):



The face groups of these two knots are identical except that the one on the left has two additional generators ***b′*** and ***d*** and two extra relations:

$$\boldsymbol{a} + \boldsymbol{d} = \boldsymbol{b} + \boldsymbol{c} \text{ and } \boldsymbol{a} + \boldsymbol{d} = \boldsymbol{b}' + \boldsymbol{c}.$$

If the Face Group of the knot on the right is written as [...| ...], the one on right would be

$$[\ldots, \boldsymbol{b}', \boldsymbol{d} \mid \ldots, \boldsymbol{a} + \boldsymbol{d} = \boldsymbol{b} + \boldsymbol{c}, \boldsymbol{a} + \boldsymbol{d} = \boldsymbol{b}' + \boldsymbol{c}]$$

where the generators and relations indicated by the dotted lines are identical for the two knots except that some of the occurrences of ***b*** in the relations of the first presentation may be replaced by ***b′*** in the second. This is because the face called ***b*** in the right-hand knot is split into two faces, ***b*** and ***b′*** in the map on the left.

Now the relations:

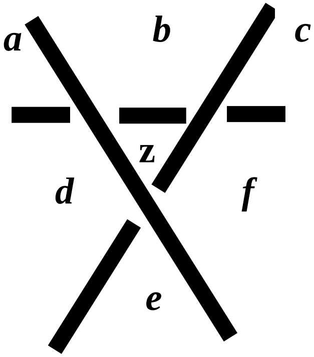
$$\left.\begin{array}{c} \boldsymbol{a} + \boldsymbol{d} = \boldsymbol{b} + \boldsymbol{c} \\ \boldsymbol{a} + \boldsymbol{d} = \boldsymbol{b}' + \boldsymbol{c} \end{array}\right\} \text{ are equivalent to } \left.\begin{array}{c} \boldsymbol{b}' = \boldsymbol{b} \\ \boldsymbol{d} = \boldsymbol{b} + \boldsymbol{c} - \boldsymbol{a} \end{array}\right\}.$$

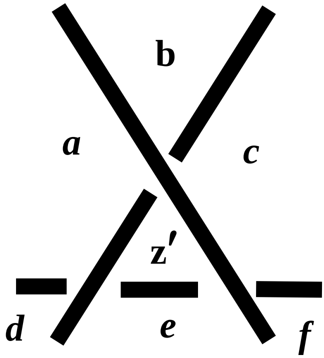
We may omit the generator ***d*** since it's only involved in the relation $\boldsymbol{d} = \boldsymbol{b} + \boldsymbol{c} - \boldsymbol{a}$.

Hence $[\ldots, \boldsymbol{b}', \boldsymbol{d} \mid \ldots, \boldsymbol{a} + \boldsymbol{d} = \boldsymbol{b} + \boldsymbol{c}, \boldsymbol{a} + \boldsymbol{d} = \boldsymbol{b}' + \boldsymbol{c}]$
$\cong [\ldots, \boldsymbol{b}' \mid \ldots, \boldsymbol{b} = \boldsymbol{b}']$.

We may now eliminate the generator ***b′*** and the relation $\boldsymbol{b} = \boldsymbol{b}'$ provided that all occurrences of ***b′*** in the other relations are replaced by ***b***. But this gives us precisely the presentation for the right-hand knot.

Type III moves don't change F(K):





For the crossings not shown the relations are identical. The remaining three relations are:

LEFT HAND-MAP	**RIGHT-HAND MAP**
$\left.\begin{aligned} a + d &= b + \mathbf{z} \\ b + \mathbf{z} &= c + f \\ \mathbf{z} + f &= d + e \end{aligned}\right\}$	$\left.\begin{aligned} a + \mathbf{z}' &= b + c \\ c + f &= \mathbf{z}' + e \\ \mathbf{z}' + e &= a + d \end{aligned}\right\}$
i.e. $\left.\begin{aligned} \mathbf{z} &= d + e - f \\ a + d &= c + f \\ a + f &= b + e \end{aligned}\right\}$	i.e. $\left.\begin{aligned} \mathbf{z}' &= b + c - a \\ a + d &= c + f \\ a + f &= b + e \end{aligned}\right\}$

Since **z** and **z′** don't appear in any other relation, and since we can express them in terms of the other generators, we may remove them from the sets of generators, together with their defining relations which are the first relation in each set of three. Now both knots have identical generators and relations.

Since equivalent knots can be obtained from one another by sequences of Reidemeister moves, it follows that the face group is an invariant of a knot. This means that if K_1 and K_2 are two knots and $F(K_1) \ncong F(K_2)$ then $K_1 \not\approx K_2$. That is, non-isomorphic Face Groups means inequivalent knots. But beware: **isomorphic Face Groups do not guarantee equivalent knots**.

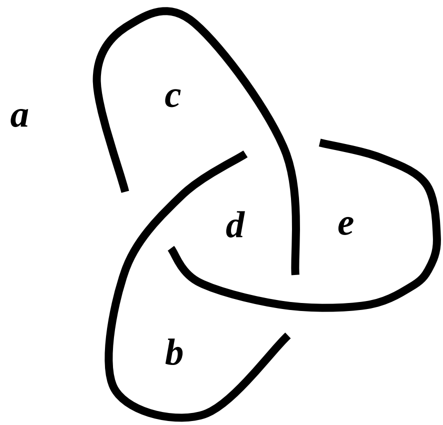
Example 2:

Suppose K_1, K_2 and K_3 are three knots. Suppose that:

$$F(K_1) \cong \mathbb{Z}_3 \oplus \mathbb{Z} \oplus \mathbb{Z};$$
$$F(K_2) \cong \mathbb{Z}_5 \oplus \mathbb{Z} \oplus \mathbb{Z};$$
$$F(K_3) \cong \mathbb{Z}_5 \oplus \mathbb{Z} \oplus \mathbb{Z};$$

Then we can conclude that $K_1 \ncong K_2$ and $K_1 \ncong K_3$. But we can't conclude that $K_2 \approx K_3$. Maybe they're equivalent — or maybe not.

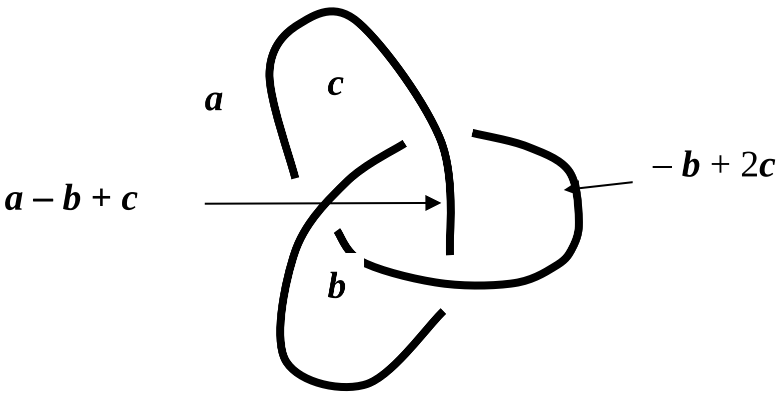
Example 3: Find the Face Group of the trefoil knot:



F(trefoil) ≈ [***a***, ***b***, ***c***, ***d***, ***e*** | ***a*** + ***c*** = ***b*** + ***d***, ***c*** + ***d*** = ***a*** + ***e***, ***d*** + ***e*** = ***a*** + ***b***]. Can you identify this group? Probably not – it's a bit difficult. Can we suggest an easier way?

Rather than create a separate generator for each face we can create three generators ***a***, ***b***, ***c*** and then express remaining faces in terms of them. (A good idea is for one of these faces to be the outside of the knot.) While ever we have crossings with three out of the four adjacent faces labelled, we can express the remaining face in terms of them. But if we should reach a situation where we only have crossings with just two, or fewer, faces labelled we must introduce a new variable. So let us repeat Example 3 incorporating this idea.

Example 3 (repeated):



At the left-hand crossing we have ***a*** + ***c*** = ***b*** + ? where ? is the middle face. Hence ? = ***a*** – ***b*** + ***c***.

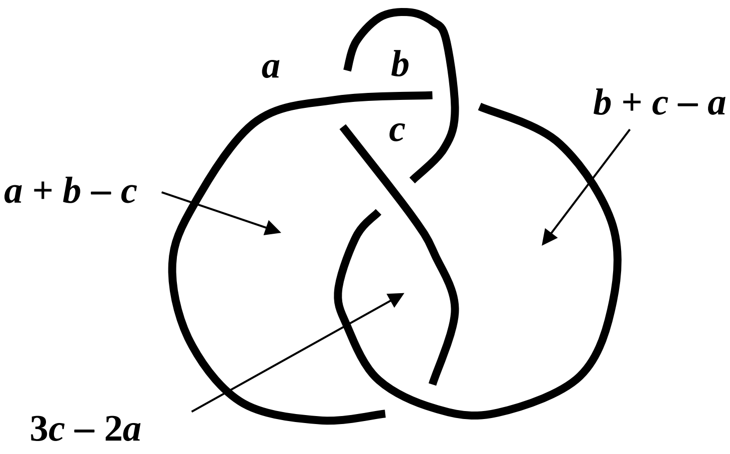
At the top crossing on the right we have
$\boldsymbol{a} + ? = \boldsymbol{c} + (\boldsymbol{a} - \boldsymbol{b} + \boldsymbol{c}) = \boldsymbol{a} - \boldsymbol{b} + 2\boldsymbol{c}$.
Hence $? = -\boldsymbol{b} + 2\boldsymbol{c}$.

All our faces are labelled but we still have one crossing to consider. At the bottom right we have

$$(\boldsymbol{a} - \boldsymbol{b} + \boldsymbol{c}) + (-\boldsymbol{b} + 2\boldsymbol{c}) = \boldsymbol{a} + \boldsymbol{b}.$$

Hence $-3\boldsymbol{b} + 3\boldsymbol{c} = 0$.
So the Face Group is $[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \mid -3\boldsymbol{b} + 3\boldsymbol{c} = \boldsymbol{0}]$.
Let $\boldsymbol{x} = \boldsymbol{b} - \boldsymbol{c}$. Then F(trefoil) $\cong [\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{x} \mid 3\boldsymbol{x} = 0]$. This is isomorphic to $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_3$.

Example 4: Find the Face Group of the Figure 8 knot.



At the last crossing we get $(3\boldsymbol{c} - 2\boldsymbol{a}) + (\boldsymbol{b} + \boldsymbol{c} - \boldsymbol{a})$
$= (\boldsymbol{a} + \boldsymbol{b} - \boldsymbol{c}) + \boldsymbol{a}$. This simplifies to $5\boldsymbol{a} = 5\boldsymbol{c}$.
Hence F(figure 8) $\cong [\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \mid 5\boldsymbol{a} = 5\boldsymbol{c}]$.
Putting $\boldsymbol{x} = \boldsymbol{a} - \boldsymbol{c}$ we can write this as $[\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{x} \mid 5\boldsymbol{x} = \boldsymbol{0}]$.

Hence F(figure 8) $\cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_5$. Since the face groups of the trefoil knot and the Figure-8 knot are not isomorphic it follows that they're inequivalent knots.

You'll have noticed that all the face groups we've obtained include two copies of $\mathbb{Z}$. It's only the finite part that distinguishes them. In fact every face group includes at least two copies of $\mathbb{Z}$.

§10.3. The Alexander Group of a Knot

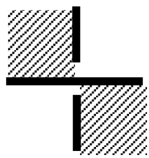
Theorem 2: If K is a knot then F(K) $\cong \mathbb{Z} \oplus \mathbb{Z} \oplus$ A(L) for some abelian group A(L).

Proof: Being finitely generated

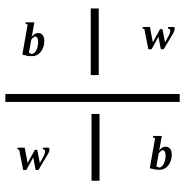
$$F(K) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \ldots \oplus \mathbb{Z} \oplus T$$

where T is finite and where there are n copies of $\mathbb{Z}$. We must show that $n \geq 2$.

Suppose you 2-colour a map of the link, black and white.



Now add additional relations which equate all the generators for the black faces to b and all the white faces to w. The relations will all collapse to $\boldsymbol{b} + \boldsymbol{w} = \boldsymbol{w} + \boldsymbol{b}$.



Hence the group will become

$$[\boldsymbol{b}, \boldsymbol{w} \mid \boldsymbol{b} + \boldsymbol{w} = \boldsymbol{w} + \boldsymbol{b}] \cong [\boldsymbol{b}, \boldsymbol{w} \mid \quad],$$

with an empty set of relations.
This is $\mathbb{Z} \oplus \mathbb{Z}$ and so $n \geq 2$.

The **Alexander Group** of a knot K is the group A(K) such that $F(K) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus A(K)$.

Example 5: A(trefoil) $\cong \mathbb{Z}_3$, A(figure 8 knot) $\cong \mathbb{Z}_5$.

Theorem 3: The Alexander Group of a knot K is finite of odd order.
Proof: By suitably swapping overpasses and underpasses at certain crossings, any knot can be transformed to the unknot. For example by changing one of the crossings in the figure 8 knot we get an unknot.

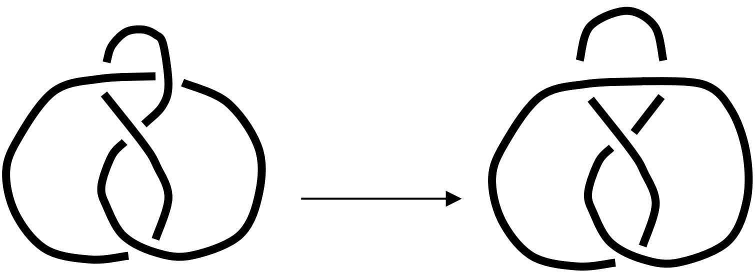


figure 8 knot **unknot**

The effect on the Face Group is to change some of the relations of the form $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{c} + \boldsymbol{d}$ into $\boldsymbol{a} + \boldsymbol{c} = \boldsymbol{b} + \boldsymbol{d}$.

This makes no difference modulo 2. The face group of the unknot is clearly $\mathbb{Z} \oplus \mathbb{Z}$, which modulo 2 becomes $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. Hence if $F(K) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \ldots \oplus \mathbb{Z} \oplus T$ where there are ($n \geq 2$) copies of $\mathbb{Z}$ and where T is finite, we must have $n = 2$ and $|T|$ must be odd.

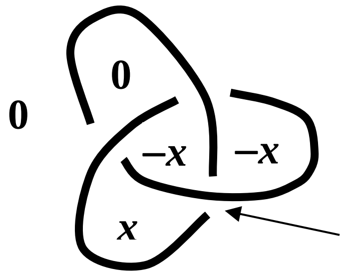
The Alexander Group of a knot is a useful invariant. However there's a fair amount of work in computing it. We can speed things up in two ways:

(1) Put the generators for two adjacent faces at some crossing to zero. This has the effect of removing the two superfluous copies of $\mathbb{Z}$ and gives just the Alexander Group. It makes things even easier if you use the outside of the link as one of these faces.

(2) Rather than label all the remaining faces with distinct labels and then writing down all the relations, simply start by using the generator $\boldsymbol{x}$ for any one of the remaining faces at the first crossing and compute the remaining face in terms of $\boldsymbol{x}$. (It will either be $\boldsymbol{x}$ or $-\boldsymbol{x}$.) Continue labelling faces around other crossings in terms of $\boldsymbol{x}$. While ever you have labels for 3 out of the 4 faces at a crossing you can use the relation for that crossing to label the 4^{th} face.

In reasonably simple cases you can express every face in terms of this one generator ***x*** and at the end there'll be just one relation, of the form $m\boldsymbol{x} = 0$, for some positive integer m, and then you can deduce that the Alexander Group of the link is $\mathbb{Z}_m$ and the Alexander Number is m.

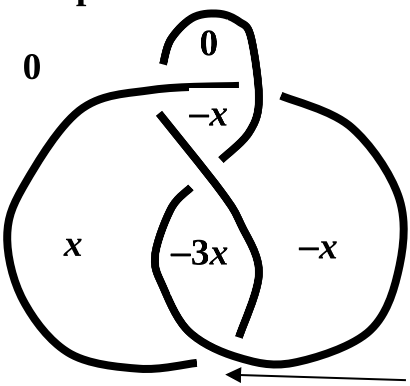
Example 6:



$A(K) \cong [x \mid 3x = 0]$
$\cong \mathbb{Z}_3$.

Here $x + 0 = -2x$

Example 7:



$A(K) \cong [x \mid 5x = 0]$
$\cong \mathbb{Z}_5$.

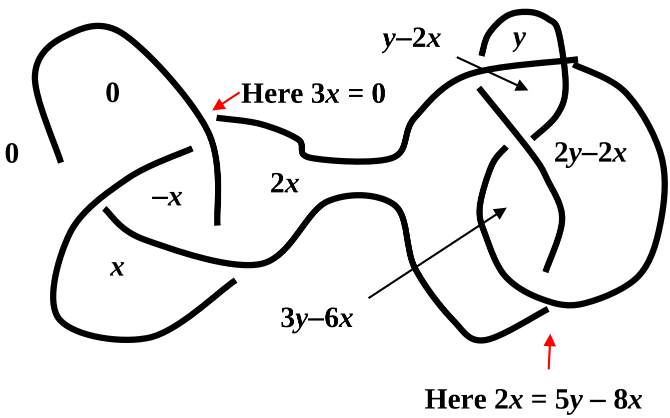
Here $x + 0 = -3x - x$

In more complicated cases you may get to a point where you reach the situation where there remain unlabelled faces, but at no crossing are three out of the four faces labelled. In this case you simply label one of them ***y*** and proceed, expressing faces in terms of ***x*** and ***y***. With a very complicated link, you might need to introduce several new variables in this way. But you'll end up with as many relations as variables.

Suppose we have more than one variable, and the same number of equations. We saw in Chapter 9 how to place the coefficients in a set of relations in a matrix and how to write the group, up to isomorphism, as a direct sum of cyclic groups using integer row and column operations:

- swap rows or columns,
- multiply a row or column by –1,
- subtract an integer multiple of one row from another or one column from another.

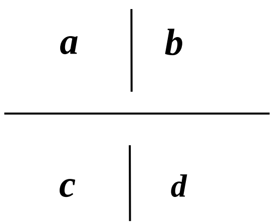
Example 8: Here is a harder example where we have more than one variable.



$$
\begin{aligned}
A(K) &\cong [\boldsymbol{x}, \boldsymbol{y} \mid 3\boldsymbol{x} = 0,\ 2\boldsymbol{x} = 5\boldsymbol{y} - 8\boldsymbol{x}] \\
&\cong [\boldsymbol{x}, \boldsymbol{y} \mid 3\boldsymbol{x} = 0,\ 10\boldsymbol{x} - 5\boldsymbol{y} = 0] \\
&\cong \begin{bmatrix} 3 & 0 \\ 10 & -5 \end{bmatrix} \\
&\cong \begin{bmatrix} 3 & 0 \\ 1 & -5 \end{bmatrix} \ R_2 - 3R_1 \\
&\cong \begin{bmatrix} 1 & -5 \\ 3 & 0 \end{bmatrix} \ R_1 \leftrightarrow R_2 \\
&\cong \begin{bmatrix} 1 & -5 \\ 0 & 15 \end{bmatrix} \ R_2 - 3R_1 \\
&\cong \begin{bmatrix} 1 & 0 \\ 0 & 15 \end{bmatrix} \ C_2 + 5C_1 \\
&\cong \mathbb{Z}_{15} \\
&\cong \mathbb{Z}_3 \oplus \mathbb{Z}_5.
\end{aligned}
$$

§10.4. Errors In Calculating Alexander Groups

(1) Going the wrong way across a crossing.



means $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{c} + \boldsymbol{d}$
not $\boldsymbol{a} + \boldsymbol{c} = \boldsymbol{b} + \boldsymbol{d}$.

(2) Starting with the combination 0, 0, $\boldsymbol{x}$, $\boldsymbol{x}$ or 0, 0, $\boldsymbol{x}$, $-\boldsymbol{x}$ depending which way you put the 0's in relation to the solid line, and then when you get stuck later on,

introducing another 0, 0, $\boldsymbol{x}$, $\boldsymbol{x}$ combination. Remember you can only do it once. It corresponds to stripping off a $\mathbb{Z} \oplus \mathbb{Z}$ from the face group. If you get stuck you must introduce a $\boldsymbol{y}$.

(3) Only discovering one equation when you have two variables. You should always get as many equations as you have variables.

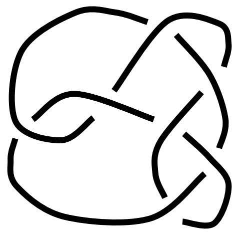
(4) Wrongly manipulating the equations. If you have more than one variable (and hence more than one equation) don't trust yourself to manipulate them. Use matrices and elementary operations.

(5) Using fractions in elementary operations or concluding that if $4\boldsymbol{x} = 2\boldsymbol{y}$ then $\boldsymbol{y} = 2\boldsymbol{x}$. Remember that you are using groups here, not real numbers.

(6) Eliminating a zero column. You may eliminate a zero row, because this is just the redundant equation 0 = 0. But a zero column represents a generator of infinite order. Of course you may eliminate a zero column provided you write $\oplus \mathbb{Z}$.

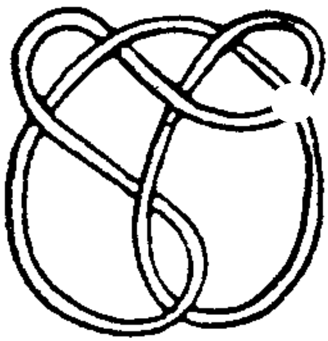
EXERCISES FOR CHAPTER 10

Exercise 1: Let K be the following knot.
(a) Draw a copy of this knot map and 2-colour it.
(b) Draw a picture of the conjugate of this knot.
(c) Draw a picture, H, of the sum of two copies of K.
(d) Make a copy of K and label the crossings + or – according as the crossing is a positive or negative crossing.



(e) Find the Alexander Group of K.
(f) Hence write down the Alexander Number of H.

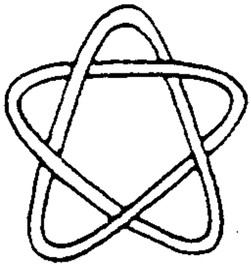
Exercise 2: Let K be the following alternating knot.
(a) Complete the drawing by showing the nature of the missing crossing.
(b) Draw a copy of this knot map and 2-colour it.
(c) Draw a picture of the conjugate of this knot.
(d) Draw a picture of the sum, H, of two copies of K.
(e) Make a copy of H and label the crossings + or – according as the crossing is a positive or negative crossing.

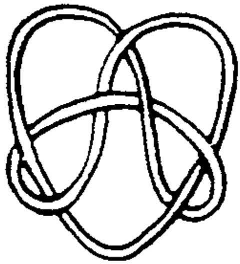


(f) Find the Alexander Group of K.
(g) Hence find the Alexander Group of H.

Exercise 3:

Find the Alexander Groups the following knots:

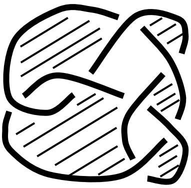




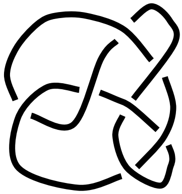
SOLUTIONS FOR CHAPTER 10

Exercise 1:

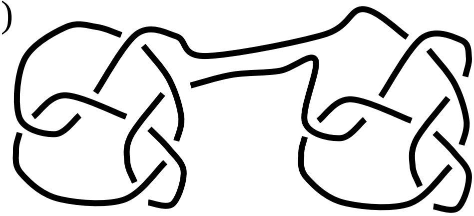
(a)



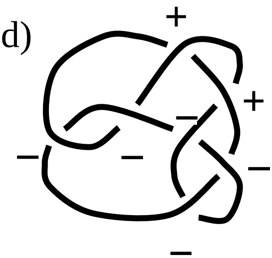
(b)



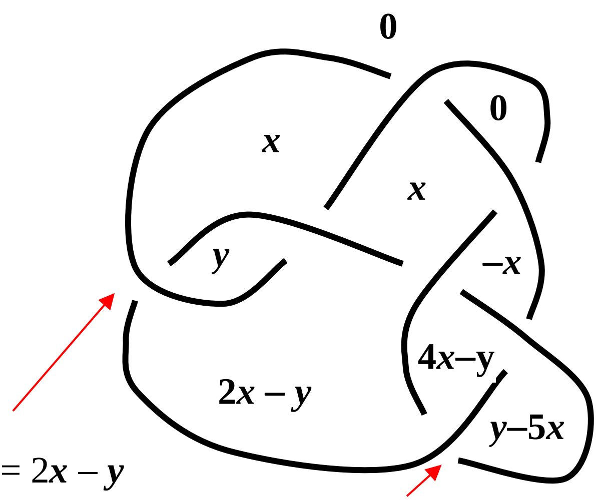
(c)



(d)



(e)



Here
$x + y = 2x - y$
i.e. $x = 2y$

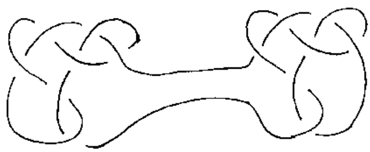
Here $6x - 2y = y - 5x$
i.e. $11x = 3y$

The Alexander Group for K $\cong \begin{bmatrix} 1 & -2 \\ 11 & -3 \end{bmatrix} \cong \begin{bmatrix} 1 & -2 \\ 0 & 19 \end{bmatrix} \cong \mathbb{Z}_{19}$.

(f) The Alexander Number for L is $19^2 = 361$.

Exercise 2:

(a), (b) (c) (d)



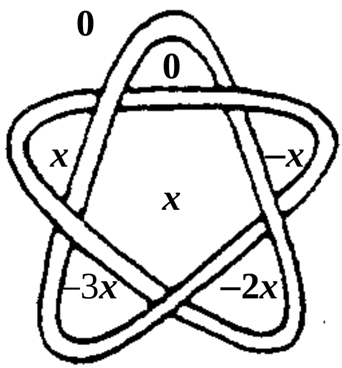
(f) The Alexander Group of K is $\begin{bmatrix} 3 & -4 \\ 7 & -3 \end{bmatrix} \cong \begin{bmatrix} 3 & -4 \\ 1 & 5 \end{bmatrix}$

$\cong \begin{bmatrix} 1 & 5 \\ 3 & -4 \end{bmatrix} \cong \begin{bmatrix} 1 & 5 \\ 0 & -19 \end{bmatrix} \cong \begin{bmatrix} 1 & 0 \\ 0 & -19 \end{bmatrix} \cong \mathbb{Z}_{19}$.

(g) The Alexander Group of L is therefore $\mathbb{Z}_{19} \oplus \mathbb{Z}_{19}$.

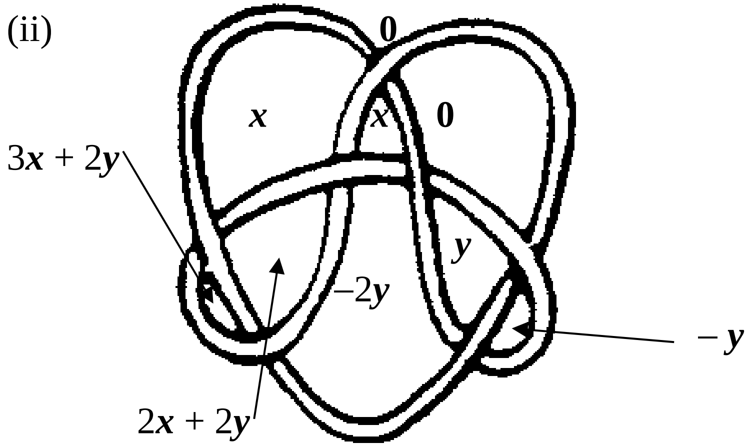
Exercise 3:

(i)



Alexander Group = $[\boldsymbol{x} \mid -3\boldsymbol{x} = 2\boldsymbol{x}] \cong [\boldsymbol{x} \mid 5\boldsymbol{x} = 0] \cong \mathbb{Z}_5$.

(ii)



Alexander Group = $[\boldsymbol{x}, \boldsymbol{y} \mid \boldsymbol{x} = 3\boldsymbol{y},\ 5\boldsymbol{x} + 6\boldsymbol{y} = \mathbf{0}]$

$$\cong \begin{bmatrix} 1 & -3 \\ 5 & 6 \end{bmatrix}$$

$$\cong \begin{bmatrix} 1 & -3 \\ 0 & 21 \end{bmatrix}$$

$$\cong \begin{bmatrix} 1 & 0 \\ 0 & 21 \end{bmatrix}$$

$$\cong \mathbb{Z}_{21}.$$